planes $z_2 = 0$ and $z_3 = 0$ cut the surface in orthogonal coincident directions.

Particular interest attaches to the tangent hyperplane perpendicular to the mean curvature **h**. This cuts the surface in the asymptotic directions and the axes of the (degenerate) conic made up of these two directions are the principal directions. The intersection of the surface and this hyperplane has therefore the fundamental properties of the Dupin indicatrix.

For the proof of the geometric results here stated and for the proofs and statements of a large number of others, many of which are entirely new, some only new statements of the results of Levi, Kommerell, or Segre, reference must be made to our complete memoir 'Differential Geometry of Two-dimensional Surfaces in Hyperspace' which will be published in the *Proceedings of the American Academy*, Boston.

[1] Kommerell, Die Krümmung der Zweidimensionalen Gebilde in ebenen Raum von vier Dimensionen, *Dissertation*, Tübingen, 1897, 53 pp; E. E. Levi, Saggio sulla Theoria delle Superficie a due Dimenzioni immersi in un Iperspazio, *Pisa, Ann. R. Scu. Norm*, **10**, 99 pp; C. Segre, Su una Classe di Superficie degl' iperspazi, *Torino, Att. R. Acc. Sci.*, **42**, 1047–1079 (1907).

[2] Ricci, *Lesioni sulla Theoria delle Superficie*, Padova, Drucker, 1898. (Lithographed, edition exhausted.)

[3] The vector analysis used is a modification of the Grassmannian system; see Lewis, *Proc. Amer. Acad. Arts Sci.*, **46**, 165–181 (1910), and Wilson and Lewis, *Ibid.*, **48**, 389–507 (1912).

[4] The corresponding indicatrix for V_2 in S_3 is not Dupin's but the range of points upon the normal described (twice) by the normal curvature vector α.

DYNAMICAL STABILITY OF AEROPLANES

By Jerome C. Hunsaker

U. S. NAVY AND MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Received by the Academy, April 6, 1916

The first rational theory of the dynamical stability of aeroplanes is due to Bryan[1] whose work was extended and applied by Bairstow[2] with wind tunnel tests on models.

The utility of such tests in predicting the aerodynamical properties of a full size aeroplane is now well understood and the validity of this application has been repeatedly demonstrated. The late E. T. Busk of the Physical Staff of the Royal Aircraft Factory, England, applied Bairstow's model tests to the design of an aeroplane and recently succeeding in perfecting an inherently stable machine which could be flown 'hands off.' Neither the details of Busk's experiments nor of the type of aeroplane developed by him have been disclosed by the British War Office.

My present investigation of two different types of aeroplanes, one a standard military tractor with no claims to inherent stability in flight, the other a machine designed to possess some degree of inherent stability while departing as little as possible from standard practice as exemplified by the other, has for its object:

(*a*) To determine the aerodynamical constants for the two aeroplanes by means of model tests in the wind tunnel.[3]

(*b*) To apply the aerodynamical constants so found in the dynamical equations of motion for the full scale aeroplanes in free flight and to examine the stability of the motion.

(*c*) To compare the stability, both lateral and longitudinal of the two chosen types of aeroplane, with a view to tracing to the individual parts of each machine their effects upon the motion.

(*d*) To attempt to formulate general qualitative conclusions which may assist constructors of aeroplanes to avoid instability or to provide any desired degree of stability.

(*e*) To throw light upon the general problem of inherent dynamical stability.

Stability distinguished as statical or dynamical.—An aeroplane in horizontal flight in still air must be driven at such speed and kept at such an inclination of wings to wind that the weight is just sustained. When in this normal attitude the aeroplane, if properly balanced, is in equilibrium. In a statical sense, this equilibrium is stable if righting moments are called into play tending to return the aeroplane to its normal attitude if by any cause it is deviated thereform. In general an aeroplane, if stable in a statical sense, will, when given an initial deviation, take up an oscillation which either may be damped out or may increase in amplitude. The aeroplane is dynamically stable if, and only if, these oscillations die out as time goes on, leaving the aeroplane in its original normal attitude. It is clear that statical stability must first be provided before the dynamical stability of a design can be examined.

The righting moments which tend to restore the aeroplane to its normal attitude are a measure of statical stability. If the statical stability is great, the period of the oscillations will be short and the motion violent, whereas an aeroplane should have a gentle motion of slow period heavily damped. An ocean liner of too great stiffness (large metacentric height) is not suitable for passenger or cattle carrying service; it is preferable in ship-design to give only enough metacentric height to insure that the vessel in never unstable and to damp the roll by generous bilge keels. In a similar manner the theory indicates that, in aeroplane-design, it is preferable to give just enough statical stability to insure that the aeroplane is never unstable and to make the oscillation dynamically stable by the use of generous damping surfaces and large wings.

The investigation is most conveniently discussed in two parts, the first dealing with the 'longitudinal' motion involving pitching of the aeroplane and rising and sinking of its center of gravity combined with change of forward speed, and the second with the 'lateral' motion, which involves side-slipping, or skidding, combined with rolling and yawing to the right and left.

Longitudinal motion.—From the dynamical equations of motion for the full scale aeroplane written down with the aerodynamical coefficients determined by tests on the models, the small oscillations about the equilibrium position are determined by three simultaneous linear differential equations with constant coefficients, of which the solution shows that the motion may be considered as composed of two oscillations one of a period of the order of 2 seconds damped to half amplitude in 0.1 seconds, the other of period from 10 to 30 seconds not very strongly damped. The short oscillation appears never to be of importance in ordinary aeroplanes, but the long oscillation, being only moderately damped, may cause trouble, especially for an aeroplane that flies with a large angle of incidence for the wings.

The calculation of the period and damping of the long oscillation was repeated for several speeds from highest to lowest corresponding to small and large angles of incidence with the results shown in the following table:

Aeroplane*	S	S	S	S	U	U	U	U
Velocity, miles-hour	76.9	53.4	44.6	36.9	79	51.8	47	44.2
Incidence of wings	0°	30°	6°	12°	1°	7°	10°	14°
Period, seconds	34.7	17.6	15.8	10.56	34.0	16.7	13.7	11.5
Damp 50 per cent, seconds	8.1	11.0	13.1	—	11.0	17.7	63.0	—
Double amplitude, seconds	—	—	—	24.7	—	—	—	24.7

* The letters S and U represent respectively the machine designed for inherent stability and a standard military tractor.

Instability at low speed.—It appears that the period becomes much more rapid at low speed, that at some critical speed the damping vanishes, and below this speed both aeroplanes become frankly unstable. This instability at extreme low speed is common to all aeroplanes and the only advantage of our 'stable' aeroplane S is that its longitudinal motion is stable down to about 40 miles per hour while aeroplane U is stable only down to about 47 miles per hour.

A study of the relative magnitudes of the coefficients for these typical aeroplanes leads to the conclusion that longitudinal instability at low speed is due, first, to the decrease in damping of the tail surfaces on account of the low speed and, secondly, to the decrease in rate of change of lifting force with change in attitude for high angles of incidence. The latter has a predominating effect on the damping of the long oscillation. Consequently, if an aeroplane is to be stable and land at a relatively slow speed, it must not operate at too great an angle of incidence. To sustain its weight it should therefore have a comparatively large wing area. The principal difference between aeroplanes S and U is that the former supports a weight of 3.55 pounds per square foot of wing area and the latter 5.2 pounds per square foot.

The following recommendations are made for an aeroplane to have its longitudinal motion damped at lower speeds than is usual in practice: (1) Provided large horizontal surfaces of long arm for damping the pitching. (2) Provide wings of such area that the slow speed does not require a great angle of incidence. Roughly the safe slow speed should not require more than 80 per cent of the maximum lift of the wings. (3) Keep the longitudinal radius of gyration small by concentrating the principal weights.

Slowness in pitching.—It may be imagined that a dynamically stable aeroplane of rapid period might be so violent in its motion that the pilot would be shaken about to such an extent as to be hindered in the performance of his military duties of observation, gun-fire, or bomb dropping. It appears that the expression representing the period of the long oscillation contains certain predominating coefficients, and a consideration of their magnitude leads to the following conclusions: The natural period of pitching is increased by: (1) High speed of flight, (2) Large damping surfaces on the tail, (3) Small angle of incidence, (4) Small righting moments.

Lateral motion.—After measuring the aerodynamical coefficients, and the radii of gyration in roll and yaw, the dynamical equations for the asymmetrical or lateral motion may be set down. For small oscillations these reduce, as in the longitudinal case, to three linear differential equations with constant coefficients. The determinant formed from the coefficients may be factored by use of approximate methods and the motion may be compounded from that represented by each of three factors.

Spiral Dive.—The first factor may correspond either to a damped or to an amplified motion. At high speeds model S shows a subsidence damped to half amplitude in 10.4 seconds. At lower speeds this damping diminishes and at 37 miles per hour the motion becomes a divergence which doubles in amplitude in 7.2 seconds. Aeroplane U is spirally unstable at high speeds. Examination of the preponderating terms in the expression representing the motion shows that the aeroplane starts off on a spiral dive.

A simple relation may be obtained involving four of the aerodynamical coefficients which, if positive, insures that spiral instability of this kind is not present. It appears that spiral instability is caused by too much fin surface to the rear or to too large a rudder, and by not enough fin surface above the center of gravity. A proper adjustment is easily obtained without sacrifice of desirable flying properties. Aeroplane S has a small rudder and wing-tips raised about 1°6; aeroplane U has no rise to wing-tips nor vertical surface above the center of gravity and has a very deep body giving the effect of a rear vertical fin. These differences in design account for the respective stability and instability of the two machines.

Rolling.—The second factor in the equation of motion represents a rolling of the aeroplane which is so heavily damped by the wide spreading wings as to be ordinarily of no consequence. In the extreme case of a 'stalled' aeroplane, the damping of the roll vanishes because the downward moving wing has no more lift than the other. Here we may expect trouble, and frequent accidents to stalled aeroplanes indicate that the pilot's lateral control by ailerons also becomes operative.

Dutch Roll.—The third element in the motion is a yawing to right and left, combined with rolling. The motion is oscillatory of period from 5 to 12 seconds, which may or may not be damped. The analogy to the 'Dutch Roll' or 'Outer Edge' in ice-skating is obvious. If the skater lean too far out on his swings he may fall, and in the same manner if the aeroplane bank too much a slight puff of wind may capsize it.

The motion of the Dutch Roll is stable provided there be sufficient vertical fin surface on the tail and not too much fin surface above the center of gravity. These requirements conflict with those previously stated for spiral stability and a compromise must be made. Over-correction of spiral instability may produce instability in the Dutch Roll and vice versa. Fortunately, the damping of rolling by the wings is helpful in both cases, and it appears possible to obtain that nice adjustment of surfaces which will render both motions stable.

Model S was stable in the Dutch Roll at all speeds, having a period from 6 to 12 seconds, and the initial amplitude damped 50 per cent in from 1.5 to 6 seconds. Model U was stable in this respect except at low speed when it showed a period of 6 seconds and the initial amplitude was doubled in 8 seconds.

The following table summarizes the results obtained for the lateral motion.

Aeroplane	S	S	S	U	U
Rise of wings	1°63	—	—	0	—
Angle of incidence	0°	6°	12°	1°	15°5
Velocity, miles	76.9	44.6	36.9	78.9	43.6
Spiral motion					
Damp 50 per cent, seconds	10.4	2.7	—	—	3.3
Double, seconds	—	—	7.2	28.0	—
Dutch roll					
Period, seconds	5.9	10.7	12.0	5.2	5.7
Damp, 50 per cent seconds	1.4	1.3	6.0	1.8	—
Double, seconds	—	—	—	—	7.7

General Conclusions.—It is believed that the majority of modern aeroplanes are spirally unstable but stable in the Dutch Roll. Furthermore it appears to be a simple matter so to adjust surfaces that any aeroplane can be made completely stable without sacrifice in speed or climb. At extreme low speed an aeroplane must be unstable in its longitudinal motion but need not be unstable laterally.

The degree of stability to provide in a given case cannot be determined from mechanical considerations alone. For example, the comfort of the pilot must be a first consideration and for this reason the righting moments giving statical stability should be small; the period of the aeroplane can then be made relatively slow, and if the damping is adequate, the free oscillations will be stable.

The theory is applied here only to flight in still air. Obviously the air is never still, and the aeroplane must finally be judged from its behavior in gusts. An inherently stable aeroplane tends to preserve its normal attitude with relation to the relative wind, and if the velocity and direction of the relative wind change in an irregular manner, the stable aeroplane will tend to follow. The result will be to force on the aeroplane a motion which will be more violent the greater the statical stability. Consequently in rough air an aeroplane very stable statically is unsuitable as a gun platform and for many other military purposes.

Considerations of theory indicate that a slight degree of statical stability combined with the maximum of damping give an aeroplane slow periods of oscillation and a dynamically stable motion, with little ill effect upon performance or controllability.[4]

[1] G. H. Bryan, *Stability in Aviation*, Macmillan, 1910.

[2] L. Bairstow, *Technical Report of the Advisory Committee for Aeronautics*, London, 1912–13.

[3] A description of the wind tunnel and the results of some experiments therein may be found in *Smithsonian Inst., Misc. Coll.*, **62**, No. 4, 1–92 (1916).

[4] Full details of this investigation will be offered for publication in a forthcoming number of *Smithsonian Misc. Coll.*